High-precision computation of two-loop Feynman diagrams with Wilson fermions

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We apply the coordinate-space method by Lüscher and Weisz to the computation of two-loop diagrams in full QCD with Wilson fermions on the lattice. The essential ingredient is the high-precision determination of mixed fermionic-bosonic propagators.

At the Lattice conference of last year we presented [1] an algebraic algorithm that allows to express every one-loop lattice integral with gluon or Wilson-fermion propagator in terms of a small number of basic constants which can be computed with arbitrary high precision [2]. This was a generalization of what we did previously with purely bosonic integrals [3] and it was also an essential step in order to apply the coordinate-space method by Lüscher and Weisz [4] to higher-loop integrals with fermions.

Let us consider, in order to fix the ideas, an example of a two-loop integral at zero external momentum, like

$$I = \int \frac{d^4l}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \frac{1}{D_F(l)D_F(r)D_F(l+r)} \qquad (1)$$

where

$$D_F(l) = \sum_{i=1}^{4} \sin^2 l_i + \frac{r_W}{4}(\hat{l}^2)^2 \qquad (2)$$

defines the usual Wilson propagator with momentum $\vec{l}$ at zero fermion mass with Wilson parameter r_W. In all our calculation we have set $r_W = 1$ but the method can be applied for any value of r_W. A possible strategy to evaluate such an integral amounts to replace each integration with a discrete sum over L points and afterwards to extrapolate to infinite L. A possibility is rewriting (1) as

$$I = \frac{1}{L^8} \sum_{l,r,l+r\neq 0} \frac{1}{D_F(l)D_F(r)D_F(l+r)} \qquad (3)$$

where each component l_i and r_i runs over the set $2\pi(n+1/2)/L$, $n = 0, \ldots, L-1$. From the sum we exclude the points such that $l + r = 0 \mod 2\pi$ where the third propagator diverges. Another possibility is to use (3) but with l_i and r_i running over the set $2\pi n/L$, $n = 0, \ldots, L-1$. In this case however there are more problems with the zero modes and one should exclude from the sum the terms with $l = r = l + r = 0 \mod 2\pi$. For this reason we have decided to use the first method. For our previous example we get for increasing values of L

$L = 10$	0.000799652
$L = 18$	0.000848862
$L = 20$	0.000853822
$L = 26$	0.000863064

Then, using an extrapolation of the form

$$a_0 + \frac{a_1 \log L + a_2}{L^2} + \frac{a_3 \log L + a_4}{L^4} \qquad (4)$$

and data with $6 \le L \le 26$, we obtain the estimate

$$I \approx 0.000879776$$

which has to be compared with what we obtained by using the coordinate-space method

$$I \approx 0.0008797779181(12)$$

Let

$$G(p,q,\vec{x}) = \int dk \frac{e^{i\vec{k}\cdot\vec{x}}}{D_F^p(k) D_B^q(k)} \tag{5}$$

where $D_B(k) = \hat{k}^2$ is the usual bosonic propagator on the lattice. In the coordinate-space approach we are interested in the evaluation of the lattice sums

$$I(p_1,q_1,\vec{a},p_2,q_2,\vec{b},p_3,q_3,\vec{c}) = \tag{6}$$
$$\sum_{\vec{x}} G(p_1,q_1,\vec{x}+\vec{a}) G(p_2,q_2,\vec{x}+\vec{b}) G(p_3,q_3,\vec{x}+\vec{c})$$

In this notation our previous example corresponds to

$$I = I(1,0,\vec{0},1,0,\vec{0},1,0,\vec{0}) \tag{7}$$

In the evaluations of these sums we make use of the following advantages:

- only four infinite lattice sums must be computed;
- the $G(p,q,\vec{x})$'s can be determined with the desired precision, for a sufficiently large domain of values of $\vec{x}$, by using our algebraic algorithm [2];
- the asymptotic expansion for large values of $|\vec{x}|$ of the $G(p,q,\vec{x})$'s is easily computed. For example

$$G(1,0,\vec{x}) = \frac{1}{\pi^2}\left[\frac{1}{4x_2} - \frac{1}{x_2^2} + \frac{2x_4}{x_2^4} - \frac{10}{x_2^3} + \right.$$
$$\left. + \frac{52}{x_2^5} + \frac{160x_4^2}{x_2^7} - \frac{192x_6}{x_2^6} + \ldots\right] \tag{8}$$

where $x_n = \sum_\mu x_\mu^n$.

Let us now consider how to compute sums of the type

$$\Sigma = \sum_\Lambda f(x) \tag{9}$$

on the lattice Λ. Of course we will not be able to sum over all the lattice. If $|x|_1 = \sum_\mu |x_\mu|$, we will perform a sum over a domain of the type $D_p = \{x \in \Lambda : |x|_1 \le p\}$. The problem is to give an estimate of the error. If $f(x)$ decreases for large $|x|$ as $1/|x|^{2k}$ we expect the sum restricted to D_p to behave as

$$\Sigma(p) = \sum_{D_p} f(x) = \Sigma + \frac{A}{p^{2k-4}} + \ldots \tag{10}$$

Thus we will estimate

$$|\Sigma - \Sigma(p)| = \frac{p}{2k-4}|\Sigma(p) - \Sigma(p-1)| \tag{11}$$

Our error formula seems to work correctly. We will use this formula to estimate the error on the integration sums. We can also define an improved estimate for Σ by

$$\Sigma \approx \Sigma(p) + \frac{p}{2k-4}(\Sigma(p) - \Sigma(p-1)) \tag{12}$$

Notice that now our error estimate is *very* conservative. The larger is k the best is the estimate. For this reason, if we know the asymptotic behaviour $Af(x)$ of the function $f(x)$ for large x it is convenient to write

$$\Sigma \approx \sum_{D_p} [f(x) - Af(x)] + \sum_\Lambda Af(x) \tag{13}$$

because the difference is decreasing faster at infinity and the sum of $Af(x)$ can be computed directly on the infinite lattice by using harmonic polynomials and ζ-functions as explained in [4]. Coming back to our preferred example, by subtracting an increasing number of terms of the asymptotic expansion, we get the estimates

$$\begin{aligned} I &= 0.0008798104043 \pm 0.0000008730034 \\ &= 0.0008797776858 \pm 0.0000000029778 \\ &= 0.0008797779227 \pm 0.0000000000410 \\ &= 0.0008797779181 \pm 0.0000000000012 \end{aligned}$$

In our work we have always used the asymptotic expansions to increase the precision of the estimates. In each case we have subtracted the asymptotic behaviour to order $1/|x|^{10}$: therefore the function which is summed over a finite lattice decays at least as $1/|x|^{12}$.

A number of checks have been performed on the table of numerical integrals that we have collected (at the moment we have a number of entries of order 10^4). In particular

- In the case in which we restrict ourselves to purely bosonic integrals we compare perfectly with the numbers given in [4].
- Because of translation invariance, for every $\vec{v}$ on the lattice

$$I(p_1, q_1, \vec{a}, p_2, q_2, \vec{b}, p_3, q_3, \vec{c}) = I(p_1, q_1, \vec{a}+\vec{v}, p_2, q_2, \vec{b}+\vec{v}, p_3, q_3, \vec{c}+\vec{v}) \quad (14)$$

- From the definition of the bosonic propagator one easily gets

$$\frac{1}{D_B^p(k)} = \frac{\sum_\mu \left(2 - e^{ik_\mu} - e^{-ik_\mu}\right)}{D_B^{p+1}(k)} \quad (15)$$

  which can be used to derive relations among different I's. Similarly, other relations can be obtained from the definition of the fermionic propagator.

- New identities are obtained by integration by parts, that is by using

$$\int dk_\mu \frac{\partial}{\partial k_\mu} F(k_\mu) = 0 \quad (16)$$

All the checks we have performed are satisfied with a precision of at least 10^{-10}

We have prepared a completely automatic procedure which evaluates Feynman diagrams at two loops. It goes through the following steps

1. Each diagram is reduced as a sum of the integrals I's, previously defined in (6). A mass is added at all propagators in order to regularize the infrared divergences.
2. All the possible symmetries (cubic: translations, permutations of the axes, inversions of the axes; permutations of the three propagators) are used to reduce the number of terms.
3. Through subtractions all the terms are written as convergent sums plus product of 1-loop integrals.
4. All 1-loop integrals are expressed as in [1,2].
5. The convergent sums are replaced by their numerical estimate obtained from a precompiled table of lattice sums in the domain D_{21}.

This procedure is now being used to compute the mixing coefficients of the four fermion operators of the lattice weak hamiltonian with the dimension 5 operators. We have chosen this computation because it starts at the two-loop level and because these coefficients have already been studied by using the momentum-space approach [5]. The computation is highly non trivial. The typical input, for each diagram, contains order 10^4 terms and produces a final result of order 10^{-4}. Unfortunately at the moment our results are still preliminary and we are performing all possible checks on our evaluations.

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